Hierarchical Approximations of a Function by Polynomials in LEMA

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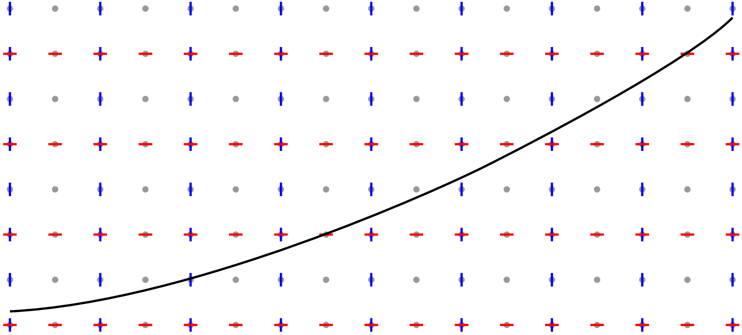
The Problem

Goal: the exhaustive test of the elementary functions for the TMD in a fixed precision (e.g., in binary64), i.e. "find all the breakpoint numbers x such that $f(x)$ is very close to a breakpoint number".

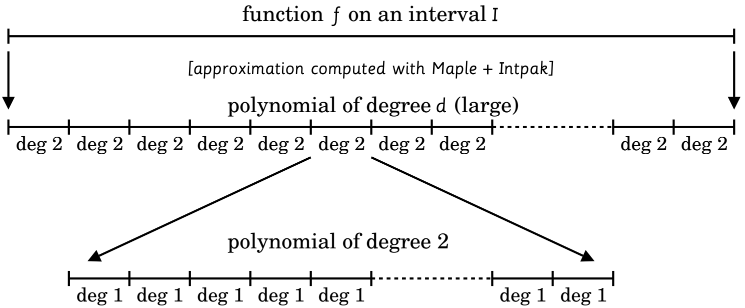
Breakpoint number: machine number or midpoint number.

$\rightarrow$ Worst cases for f and the inverse function f^{-1}.



Hierarchical Approximations by Polynomials

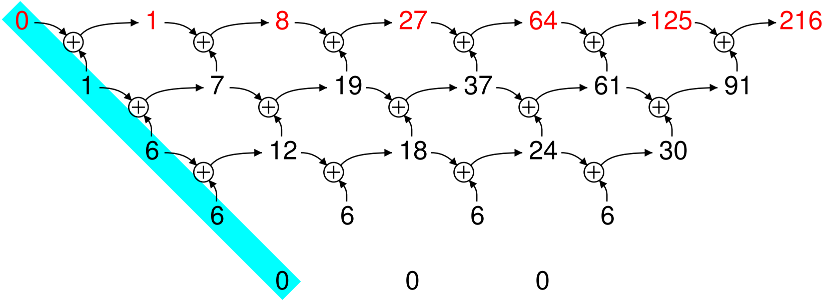
Current implementation (but one could have more than 3 levels):



- Finding approximations must be very fast: from the previous one.
- Degree-1 polynomials: fast algorithm that computes a lower bound on the distance between a segment and $\mathbb{Z}^2$ (in fact, this distance, but on a larger domain) [filter] + slower algorithms when needed.

Computing the Successive Values of a Polynomial

Example: $P(X) = X^3$. Difference table:



On the left: coefficients of the polynomial in the basis

$$\left\{1, X, \frac{X(X-1)}{2}, \frac{X(X-1)(X-2)}{3!}, \ldots\right\}$$

Representation in the LEMA Tree

Computations can (and will) be done modulo some constant (much faster).
$\rightarrow$ The corresponding arithmetic must be supported by LEMA.

In practice, some coefficients will be close to 0 (either from above or from below).
$\rightarrow$ In the LEMA tree, notion of magnitude (like with real numbers).

How can this be expressed in LEMA?

- With a list (tuple) containing the coefficients? (But the degree d is not necessarily a constant parameter.)
- With a function taking two arguments i and n returning the coefficient $a_i(n)$ of $P(X+n)$ in the basis

$$\left\{1, X, \frac{X(X-1)}{2}, \frac{X(X-1)(X-2)}{3!}, \ldots\right\}?$$

The polynomial object is less visible, but this should be easier.

An Example of Coefficient Values

An example of coefficient values from the current implementation:

```
a0_0  =  A6ABF7160809CF4F 3C762E7160F38B4E
a0_1  =  5458A4173B436123 9CA0E833FEB6CB85 ABFCA8C9
a0_2  =  000002B7E1516295 CCAFB049B66C0BEA 354AA25BAAB8404F
a0_3  =  000000000000000A DF85458A6CF1C94C 3BA51465E493E36F D8B90AB5
a0_4  =  0000000000000000 000000002B7E1516 2A0AC34F5D426FDA C4D9DF953D0EDFFB 16FE1543
a0_5  =  0000000000000000 0000000000000000 00ADF85458A986FD E62637A70A321BD8 4F1A4229E540A478
a0_6  =  0000000000000000 0000000000000000 000000000002B7E1 51628AED2A6ABF71 58809CF4F3C762E7

a1_0  =  5BF0A8B145769AA5 225B715628DDCEBF
a1_1  =  00000000056FC2A2 C520B9EFDA13A7F9 8A29425F
a1_2  =  0000000000000000 0015BF0A8B14AE65 D47E77DFB318E888
a1_3  =  0000000000000000 00000000000056FC 2A2C53678FA65285 D9F0CD61
a1_4  =  0000000000000000 0000000000000000 0000015BF0A8B150 561FEAAC8725FFD3 CD14C497
a1_5  =  0000000000000000 0000000000000000 0000000000000005 6FC2A2C515DA54D5 7EE2B10139E9E78F

a2_0  =  0000000000000ADF 85458A2BB500A728
a2_1  =  0000000000000000 0000002B7E151629 05D02CC9
a2_2  =  0000000000000000 0000000000000000 ADF85458A573315C
a2_3  =  0000000000000000 0000000000000000 0000000002B7E151 629B3C88
a2_4  =  0000000000000000 0000000000000000 0000000000000000 000ADF85458A2BB4 A9AAFDC5
```

From an Interval to the Next One

The problem (to be considered recursively):

- Input: a polynomial P of degree d on an interval I.
- The interval I is split into subintervals J_n of the same length.
- The polynomial P will be approximated by polynomials P_n of degree d' on the intervals J_n (sequentially).
- Goal: generate code to compute the (initial) coefficients of P_n very quickly (from the work done for P_{n-1} on J_{n-1}).
- All errors need to be bounded formally: an acceptable error bound will be part of the input, and various parameters (the precision of the coefficients, etc.) will be determined from it.

From an Interval to the Next One [2]

Two methods:

1. Take into account the computations that haven't been done, i.e. those involving the coefficients of degrees $> d'$.
   $\rightarrow$ Linear combinations of coefficients: additions and multiplications of coefficients by integer constants (constant in the generated code).
2. Use the fact that the intervals J_n have the same length: each (initial) degree-i coefficient of P_n can be seen as the value of a polynomial $a_i(n)$.
   $\rightarrow$ The *difference table method* can be used: only additions.

Error Bounds

Three kinds of errors:

- Error due to the approximation of function f by a polynomial.
- Approximation errors: coefficients of degree $> d'$ are ignored.

  $\rightarrow$ Error bound of the form: $\displaystyle\sum_{i=d'+1}^{d} U_i \cdot |a_i(0)|$

  where U_i depends on i and the size of the intervals I and J_n.
- Rounding errors on the coefficients a_i (due to their representation with an absolute precision n_i): initial errors and after each computation.

  $\rightarrow$ Error bound on $a_i(m)$ of the form: $\displaystyle\sum_{j=i}^{d'-1} V_{i,j,m} \cdot 2^{-n_j}$.

Formally determining U_i and $V_{i,j,m}$ can be to difficult to be done automatically. But it should be possible to verify (prove) them with LEMA with conventional error analysis:

- with help of computer algebra software for generic formulas;
- numerically, after instantiation.

Distribution of the Jobs on Different CPU's

2 possibilities:

- Completely independent jobs (as with the current implementation): the domain is split into intervals I, on which f is approximated by a polynomial P and so on. If need be, the code generation can be done on a different machine.
- On some machine (regarded as a server), f is approximated by P on an interval I, which is split into N subintervals J_n; the coefficients of P_n are computed directly. The corresponding N jobs are distributed on different machines.

Note: the input parameters can be chosen to control the size thus the estimated average execution time of a job (actually the order of magnitude).

LEMA Features That Will Be Needed

- Fast automatic generation of correct (in fact, proved) code, possibly with annotations (for provers, but this is currently limited because of the specific arithmetic).
- Possibility to test various parameters.
- Code instrumentation (was forgotten in most discussions), e.g. to count the number of word additions. For instance, transform the LEMA tree to replace a result x by a pair (x, c_x), and $x + y$ by $(x + y, c_x + c_y + 1)$?
- Checking that the LEMA tree is correct, e.g. that formulas written by the human are correct.